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ACADEMIC PRESS
Journal of Sound and Vibration 265 (2003) 451–458
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Letter to the Editor

Forced non-linear vibrations of a symmetrical two-mass system

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Received 18 July 2002; accepted 5 November 2002

1. Introduction

In Refs. [1–5] the free vibrations of a two-mass system connected to each other and with the fixed ends are investigated. The non-linearity is of the cubic type. The model is a system of two non-linear differential equations where the solution is in the form of Jacobi elliptic functions. In discussing the solution obtained, the influence of the non-linear properties of the connecting spring on the motion is shown.

The forced vibrations of a simple two-mass system containing only two connected masses is discussed in Ref. [6]. A connecting spring with quadratic non-linear properties is considered. For the case when the amplitude of the force on the leading element is constant, the vibration properties of the system deeply depend not only on the coefficient of non-linearity of the spring but also on the amplitude of the force. The motion of the leading and the lead mass is a function of the mass distribution of the system.

The aim of this paper is to analyze the forced vibrations of a symmetric two-mass system connected to fixed supports with linear springs. The connecting spring between the masses has strong non-linear elastic properties. The mathematical model is a system of two coupled non-linear and non-homogenous differential equations. The exact solution of the system is in the form of Jacobi elliptic functions. The vibration motion of the leading and the lead mass under influence of the constant force is investigated. The obtained solutions give some practical suggestions for the constructions of machine tools.

2. Model of the system

The symmetrical two-mass system contains two equal masses m, which are connected to each other with a non-linear elastic spring (Fig. 1). The non-linearity is of the quadratic type. The

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0022-460X/03/$ - see front matter
doi:10.1016/S0022-460X(02)01569-9

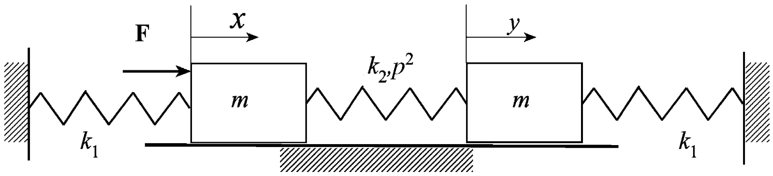

Fig. 1. The model of the system.

masses are connected to fixed supports with symmetric linear springs with rigidity coefficient k_1. A constant force F acts on the leading mass. The kinetic energy and the potential energy of the system are

$$T = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2),$$
$$V = \tfrac{1}{2} k_1(x^2 + y^2) + \tfrac{1}{2} k_2(x - y)^2 + \tfrac{1}{3} p^2(\pm)|x - y|(x - y)^2, \tag{1}$$

where x and y are the deflections of the masses, and k_2 and p^2 are the linear and the non-linear coefficients of rigidity of the connecting spring. In the relation for the potential energy the *absolute* function exists. It is due to the fact that the force in the spring is a quadratic function of deformation and it satisfies the antisymmetric condition. Namely, if the spring is extended, a force appears which has the tendency to contract and to put the spring in the previous state. The same happens when the spring is pressed. In the first case the deformation is positive, and in the second case it is negative. The signs in the bracket, $(\pm)$, indicate the hard and the soft spring, respectively, i.e., the plus sign in the bracket is for a hard spring and the minus sign in the bracket is for a soft spring. This meaning of the bracket $(\pm)$ is applied throughout the whole paper.

Using relations (1) and the fact that the force F acts, the mathematical model of the system is

$$m\ddot{x} + k_1 x + k_2(x - y) + p^2(\pm)|x - y|(x - y) = F,$$
$$m\ddot{y} + k_1 y - k_2(x - y) - p^2(\pm)|x - y|(x - y) = 0 \tag{2}$$

subject to the initial conditions

$$x(0) = y(0) = 0, \quad \dot{x}(0) = \dot{y}(0) = 0. \tag{3}$$

3. Solving procedure

Introducing the new variables

$$X = x - y,$$
$$Y = x + y, \tag{4}$$

in Eqs. (2), the transformed equations of motion are

$$\ddot{Y} + \Omega^2 Y = a, \tag{5}$$

$$\ddot{X} + cX + (\pm)b^2|X|X = a, \tag{6}$$

where

$$c = \frac{(k_1 + 2k_2)}{m} = \Omega^2 + \frac{2k_2}{m}, \quad b^2 = \frac{2p^2}{m},$$
$$\Omega^2 = \frac{k_1}{m}, \quad a = \frac{F}{m}. \tag{7}$$

The initial conditions (3) are transformed to

$$Y(0) = 0, \quad \dot{Y}(0) = 0, \tag{8}$$

$$X(0) = 0, \quad \dot{X}(0) = 0. \tag{9}$$

Eqs. (5) and (6) represent two separate second order non-homogenous differential equations which are solved separately subject to the initial conditions (8) and (9), respectively. Both differential equations describe the forced vibrations of a system with two degrees of freedom, where the first system is linear, and the second is non-linear.

Substituting the solutions of Eqs. (5) and (6) into Eq. (4), the motion of the leading and the lead masses are

$$x = \frac{Y + X}{2}, \quad y = \frac{Y - X}{2}. \tag{10}$$

It means that the total motion of the masses is a simple superposition of two separate oscillatory motions.

For discussion of results (10), it is necessary to obtain the exact solutions of Eqs. (5) and (6).

4. Results and discussion

Consider the solution of Eq. (5) subject to Eq. (8). It is of the oscillatory type and has the following form:

$$Y = \frac{a}{\Omega^2}(1 - \cos \Omega t) = \frac{2a}{\Omega^2} \sin^2 \frac{\Omega t}{2}. \tag{11}$$

The amplitude of vibration is a function of the external force F. The frequency and the period of vibration depend on the mass m and the rigidity k_1 of the linear spring. Namely, the mathematical model (5) describes the forced vibration of the mass m connected to a fixed support with the linear spring k_1. In this differential equation the influence of the connecting spring does not appear. It is worth saying that the function Y (11), which represents the closed-form analytical solution of Eqs. (5) and (8), depends on the square of the circular sine function and due to this fact it is a non-negative function for all values of time t.

The second equation (6) is a non-linear non-homogenous second order differential equation. In Ref. [7] the solution of the differential equation (6) with respect to the initial conditions (9) is obtained. Independently of the type of the non-linearity, i.e., for a soft and for a hard non-linearity, the solution has the square form of one of the Jacobi elliptic functions [8]. Due to this fact, the function X is also non-negative for all values of time t. In the next section the exact solution of Eq. (6) for soft, hard and linear springs are discussed.

4.1. System with soft spring

For a mechanical system with a soft spring, the lower sign in Eq. (6) is used and its particular solution for the initial conditions (9) is according to Ref. [7]

$$X = A_s\, sn^2\left(\frac{\Omega_s}{2}t, k_s^2\right), \tag{12}$$

where

$$A_s = \frac{3c}{4b^2}\left[1 - \sqrt{1 - \frac{16}{3}\frac{ab^2}{c^2}}\right], \tag{13}$$

$$\Omega_s = 2\sqrt{\frac{a}{2A_s}}, \tag{14}$$

$$k_s^2 = \frac{b^2 A_s^2}{3a}, \tag{15}$$

and *sn* is a Jacobi elliptic function (see Ref. [8]). Solution (12) represents an oscillatory motion. Introducing parameters (7), the amplitude of vibration is

$$A_s = A_l \frac{2}{1 + \sqrt{1 - \frac{16}{3}fB}}, \tag{16}$$

where the coefficients of non-linearity B is

$$B = \frac{2p^2}{(k_1 + 2k_2)}, \tag{17}$$

the forcing term is

$$f = \frac{F}{(k_1 + 2k_2)} \tag{18}$$

and

$$A_l = 2f. \tag{19}$$

Relation (19) is the amplitude of vibration of the linear system. Namely, for the linear system the non-linearity is zero, i.e., $B = 0$ and the amplitude of vibration (16) transforms to the linear one (19). Due to the non-linearity a correction to the amplitude of vibration of the linear system exists. This correction is a function not only of the forcing term but also of the coefficient of non-linearity. The amplitude of vibration is higher than for the linear system ($A_s > A_l$). For the same value of the forcing term the amplitude of vibration is higher for higher value of the coefficient of non-linearity.

The frequency of the Jacobi elliptic function is

$$\Omega_s = \frac{1}{\sqrt{2}}\Omega_l\sqrt{1 + \sqrt{1 - \frac{16}{3}fB}}, \tag{20}$$

where the frequency of the system with linear spring is

$$\Omega_l = \sqrt{\frac{k_1 + 2k_2}{m}}. \tag{21}$$

It can be concluded that the frequency of vibration of the linear system is multiplied by a correction term which is a function of the coefficient of non-linearity and forcing. For the system with a soft spring the frequency of vibration is smaller than for the linear spring. It is a function of the forcing term: the smaller the forcing term the frequency tends to Ω_l. The minimal value of the frequency is $\Omega_s = \Omega_l/\sqrt{2}$ for the forcing term $f = 3/(16B)$.

The modulus of the Jacobi elliptic function is

$$k_s^2 = \frac{16}{3}\frac{Bf}{[1+\sqrt{1-\frac{16}{3}Bf}]^2}. \tag{22}$$

For the linear case, i.e., $B = 0$, it is $k^2 = 0$ and the *sn* Jacobi elliptic function transforms to the harmonic sine function [9]. The transformed solution (12) is for the linear system

$$X = f(1 - \cos\Omega_l t) = A_l \sin^2\frac{\Omega_l}{2}t. \tag{23}$$

The period of vibration is

$$T_s = \frac{4K(k_s^2)}{\Omega_s}, \tag{24}$$

where $K(k_s^2)$ is the complete elliptic integral of the first kind [9]. Due to relations (20) and (22) it is evident that the period of vibration of the system is a function of the coefficient of non-linearity and also of the coefficient of forcing. It is not the case for the linear system (13). As the value of $4K(k_s^2) \geqslant 2\pi$ and $\Omega_s \leqslant \Omega_l$ the period of vibration of the system with soft spring is

$$T_s \geqslant T_l,$$

where $T_l = 2\pi/\Omega_l$ is the period of vibration of the system with linear spring.

The oscillatory solution (12) has a limitation: it is valid only for

$$1 - \frac{16}{3}\frac{ab^2}{c^2} \geqslant 0, \tag{25}$$

i.e.,

$$fB \leqslant \tfrac{3}{16}. \tag{26}$$

It means that the region of oscillatory motion depends on the non-linear properties of the system. The higher the non-linearity the smaller the available forcing term.

4.2. System with hard spring

For the system with a hard spring, the upper sign in Eq. (6) is used. As a consequence of the imaginary modulus transformation, the particular solution of Eq. (6) for Eq. (9) is according to Ref. [7]

$$X = A_h^* sd^2\left(\frac{\Omega_h}{2}t, k_h^2\right), \tag{27}$$

where sd is a Jacobi elliptic function (see Ref. [8]),

$$\Omega_h = 2\sqrt{\frac{a}{2A_h}\left(1 + \frac{b^2A_h^2}{3a}\right)}, \tag{28}$$

$$k_h^2 = \frac{b^2A_h^2/(3a)}{(1 + b^2A_h^2/(3a))}, \tag{29}$$

$$A_h^* = A_h\frac{1}{(1 + b^2A_h^2/(3a))} \tag{30}$$

and

$$A_h = \frac{3}{4}\frac{c}{b^2}\left(\sqrt{1 + \frac{16}{3}\frac{b^2a}{c^2}} - 1\right). \tag{31}$$

After substituting Eq. (7) into Eq. (31) it is

$$A_h = A_l\frac{2}{1 + \sqrt{1 + \frac{16}{3}Bf}}. \tag{32}$$

Solution (27) is of oscillatory type. The amplitude of vibration is

$$A_h^* = A_l\frac{1}{\sqrt{1 + \frac{16}{3}Bf}} \tag{33}$$

and it is a product of the amplitude of vibration of the linear system and a correction term which depends on the coefficient of non-linearity and the forcing term. The amplitude of vibration of the non-linear system with hard spring is smaller than the amplitude of vibration of the system with linear spring ($A_h^* < A_l$).

The frequency of vibration is

$$\Omega_h = \Omega_l\sqrt[4]{1 + \frac{16}{3}Bf}$$

which is higher than the frequency of the linear system and depends on the non-linear properties of the system and on the forcing term.

The modulus of the Jacobi elliptic function is

$$k_h^2 = \frac{\frac{16}{3}fB}{(1 + \sqrt{1 + \frac{16}{3}Bf})^2 + \frac{16}{3}fB}.$$

The period of vibration is

$$T_h = \frac{4K(k_h^2)}{\Omega_h}$$

and comparing it to the period of vibration of the system with a linear spring is

$$T_h < T_l.$$

5. Conclusions

In this paper the forced vibrations of a symmetric two-mass system connected by a spring with quadratic non-linear elastic properties is investigated. Analyzing the results obtained the following is concluded:

1. The motion of the symmetric two-mass system connected to fixed supports and subject to a constant force is oscillatory and the motion of the masses is a simple superposition of two oscillatory motions.

2. The motion of the leading mass on which the force acts and of the lead mass without forcing is different. The difference of the motion depends on the rigidity properties of the connecting spring. Namely, for higher values of the coefficient of linear rigidity k_2 the amplitude of one of the constituting motions (X) is quite small and the corresponding period of vibration tends to zero. Then the motion of both the masses are in the same direction and are approximately the same, i.e., $x \approx y \approx Y/2$. For that case, if the rigidity coefficient k_1 of the spring is also high, the vibrations of the masses are negligible. Then the active force is used for useful work and the vibrations, which are the side effect, are eliminated. This property of the system is of special interest for some machine parts: for example, in machine tools for plastic deformation of metallic and non-metallic objects where the working velocity is high [10].

3. For the special instants of time when $t = T_s$, $t = T_l$ and $t = T_h$, respectively, for the system with soft, linear and hard spring, the corresponding amplitude of vibration X is zero and the both masses have the same deflections $x(T) = y(T) = Y(T)/2$.

4. The qualitative properties of the motion of the symmetric two-mass system connected with linear or non-linear springs are the same, but differ quantitatively. For the system with a non-linear spring the frequency of vibration and also the period of vibration depend on the intensity of the external force. It is not the case for the linear spring.

Acknowledgements

The present study has been partly sponsored by Ministry of Science, Technologies and Development, Republic of Serbia (Project No. 1874).

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